Chapter 8.
Root Locus Techniques

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Introduction

- Root locus: tool of analysis and design for stability and transient response
- The root locus can be used to describe qualitatively the performance of various parameters which are changing.
  - The effect of varying gain upon percent overshoot, settling time, and peak time can be vividly displayed.
- Besides transient response, the root locus also gives a graphical representation of a system's stability.
  - Ranges of stability, ranges of instability, and the conditions that cause a system to break into oscillation.

Defining the Root Locus

- A video camera system consists of the tracking system of a dual sensor and a transmitter.
- Unbalance between the outputs of the two sensors receiving data from the transmitter causes the system to the balance out the difference and seek the source of energy.
- Analysis and design using the effect of loop gain upon the system's transient response and stability.

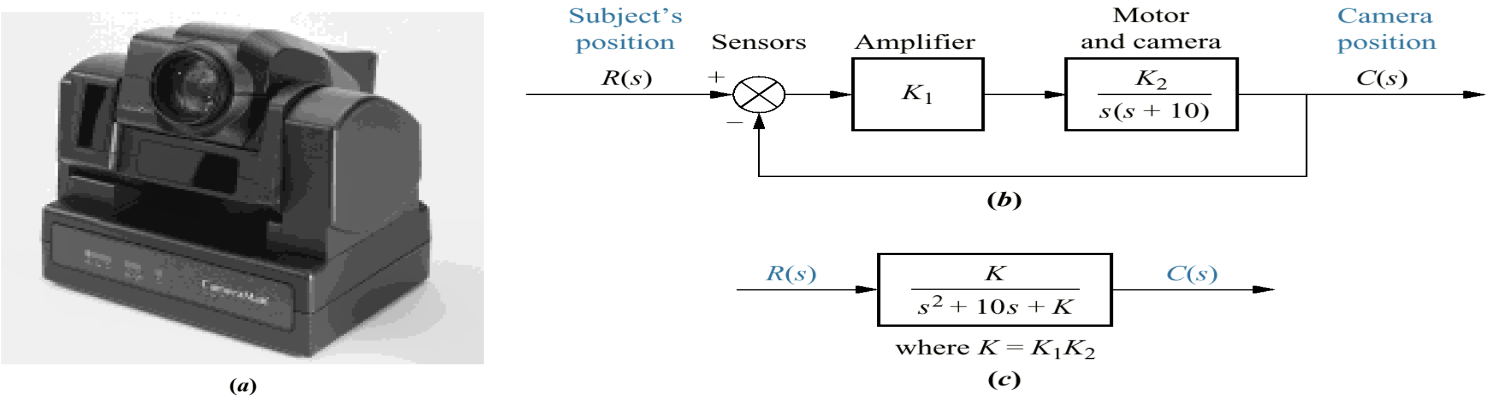

Figure 8.1
a. Presenter Camera System automatically. b. block diagram. c. closed-loop transfer function.

Defining the Root Locus

- Representation of the closed-loop poles path as the gain K.
- The root locus gain K represents a change in the transient response.

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	−5 + j2.24	−5 − j2.24
35	−5 + j3.16	−5 − j3.16
40	−5 + j3.87	−5 − j3.87
45	−5 + j4.47	−5 − j4.47
50	−5 + j5	−5 − j5

Table 8.1
Pole location as a function of gain for the system of Figure 8.4

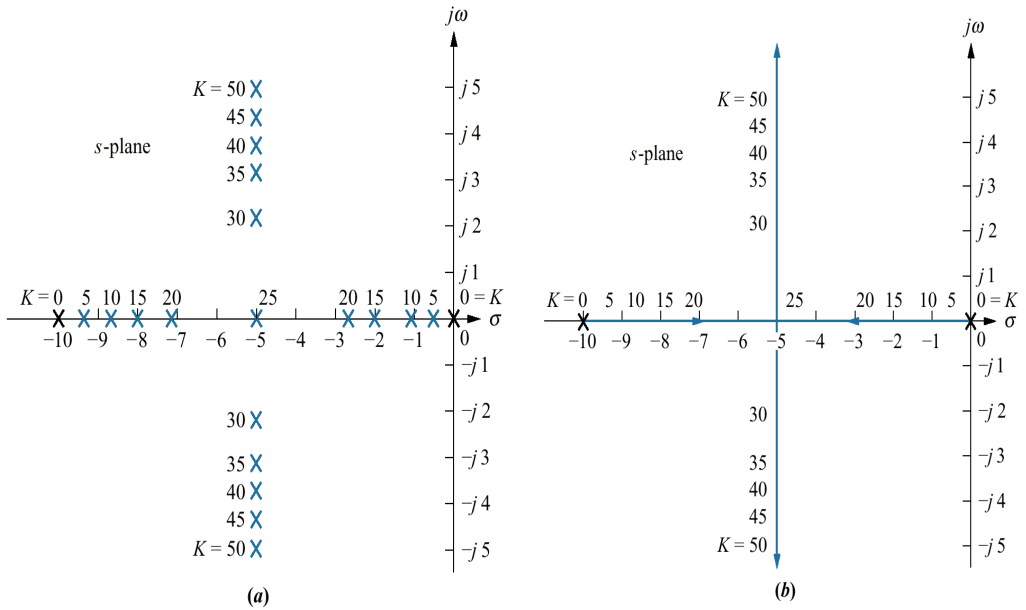

Figure 8.2
a. Pole plot from Table 8.1;
b. root locus

Properties of the Root Locus

- Sketch of the root locus for higher-order systems without considering the denominator of the closed-loop transfer function.

- Open-loop transfer function

$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)} \qquad (8.1)$$

- Closed-loop transfer function

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)} \qquad (8.2)$$

$$= \frac{KG(s)H(s)}{1+KG(s)H(s)}$$

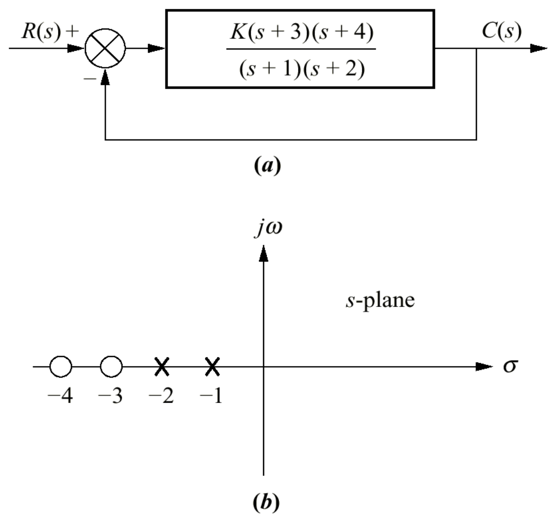

Figure 8.3
a. Example system;
b. pole-zero plot of G (s)

Properties of the Root Locus

- Consider the point $-2+j3$. If this point is a closed-loop pole for some value of gain, then the angles of the zeros minus the angles of the poles must be an odd multiple of 180° from Fig. 8.7

$$\theta_1+\theta_2-\theta_3-\theta_4=56.31^\circ+71.57^\circ-90^\circ-108.43^\circ=-70.55^\circ \quad (8.3)$$

- Gain, K

$$K=\frac{1}{|G(s)H(s)|}=\frac{1}{M}=\frac{\prod pole\ lengths}{\prod zero\ lengths} \quad (8.4)$$

- The gain value apply to $-2+j(\sqrt{2}/2)$ of Fig. 8.7

$$K=\frac{L_3L_4}{L_1L_2}=\frac{\frac{\sqrt{2}}{2}(1.22)}{(2.12)(1.22)}=0.33 \quad (8.5)$$

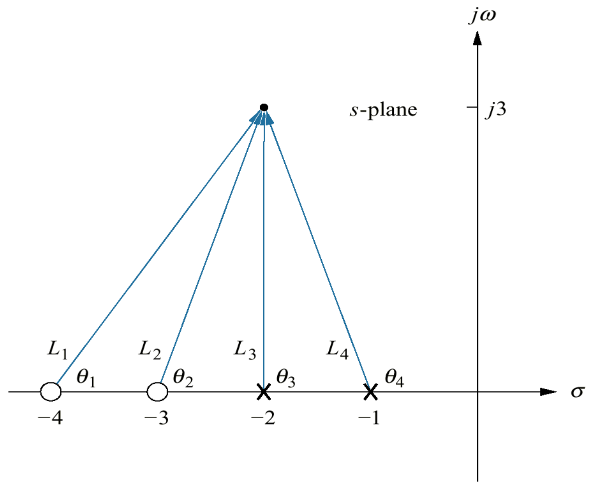

Figure 8.7
Vector representation of G(s) from Figure 8.6(a) at -2+ j 3

- The point $-2+j(\sqrt{2}/2)$ is a point on the root locus for a gain of 0.33.

SKETCHING THE ROOT LOCUS

- Once a sketch is obtained, it is possible to accurately plot just those points that are of interest to us for a particular problem.
- The following five rules allow us to sketch the root locus using minimal calculations.

1. Number of branches
   - Each closed-loop pole moves as the gain K is varied.
   - Define a branch as the path that one pole traverses
   - **The number of branches of the root locus equals the number of closed-loop poles.**
2. Symmetry
   - If complex closed-loop poles do not exist in conjugate pairs, the resulting polynomial, formed by multiplying the factors containing the closed-loop poles, would have complex coefficients-not physically realizable.
   - **The root locus is symmetrical about the real axis.**

SKETCHING THE ROOT LOCUS

3. Real-axis segments
   - The contribution to the angle at any of the points comes from **open-loop, real-axis poles and zeros that exist to the RIGHT of the respective point**
   - **On the real-axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros on the right.**

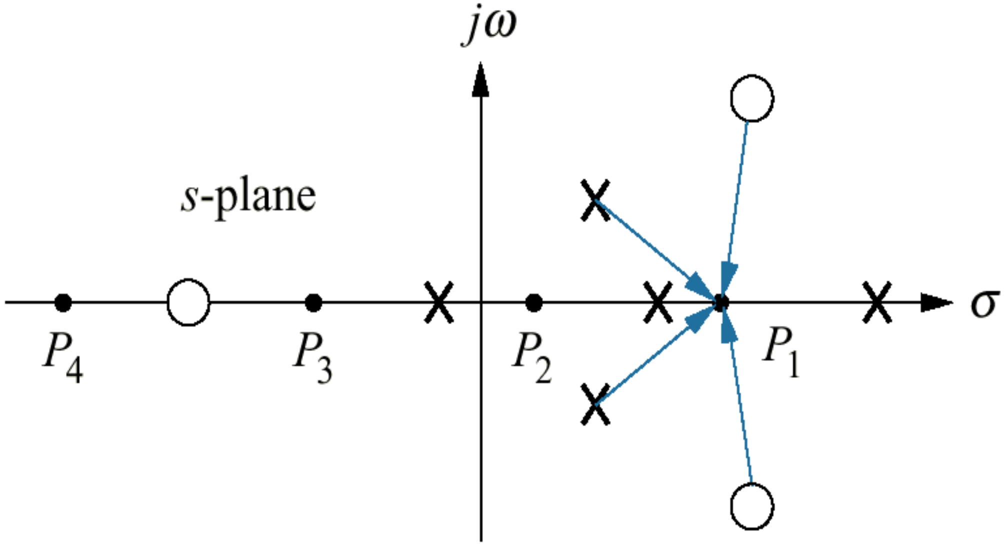

Figure 8.5
Poles and zeros of a general open-loop system with test points, P_i, on the real axis

SKETCHING THE ROOT LOCUS

4. Starting and ending points
   - As K approaches zero,

$$T(s) \approx \frac{KN_G(s)N_H(s)}{D_G(s)D_H(s) + \in} \qquad (8.6)$$

   - At high gains, where K is approaching infinity

$$T(s) \approx \frac{KN_G(s)N_H(s)}{KN_G(s)N_H(s) + \in} \qquad (8.7)$$

   - **The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.**

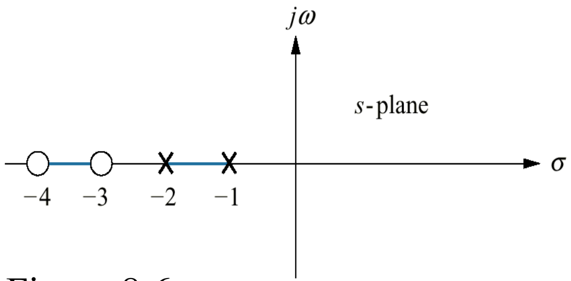

Figure 8.6
Real-axis segments of the root locus for the system of Figure 8.3

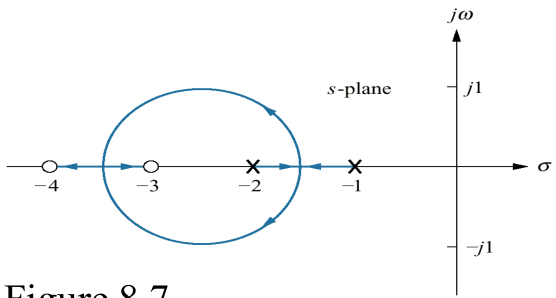

Figure 8.7
Complete root locus for the system of Figure 8.3

SKETCHING THE ROOT LOCUS

5. Behavior at infinity

- **Locate poles at infinity** for functions containing more finite zeros than finite poles.
- **The root locus approaches straight lines as the locus approaches infinity.**
- Equation of the asymptotes is given by the real-axis intercept, σ_a and angle, θ_a as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \tag{8.8}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \tag{8.9}$$

θ_a the angle is given in radians with respect to the positive extension of the real axis

- Notice that the running index, k, in Eq. (8.9) yields a multiplicity of lines that account for the many branches of a root locus that approaches infinity.

SKETCHING THE ROOT LOCUS

- Example 8.1
  - Sketching the a root locus for the system shown in Figure 8.8

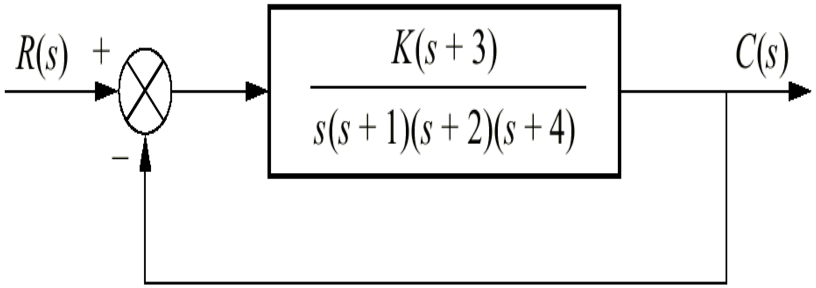

Figure 8.8
System for Example 8.1

- Using Eq.(8.8), the real-axis intercept is evaluated as

$$\theta_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3} \qquad (8.10)$$

SKETCHING THE ROOT LOCUS

- The angles of the lines that intersect at -4/3, given by Eq.(8.9)

$$\theta_a = \frac{(2k+1)\pi}{\#\ finite\ poles - \#\ finite\ zeros} \quad (8.11)$$

$$= \pi/3 \qquad for\ k = 0$$

$$= 3 \qquad for\ k = 1$$

$$= 5\pi/3 \qquad for\ k = 2$$

- Figure 8.9 shows the complete root locus as well as the asymptotes that were just calculated

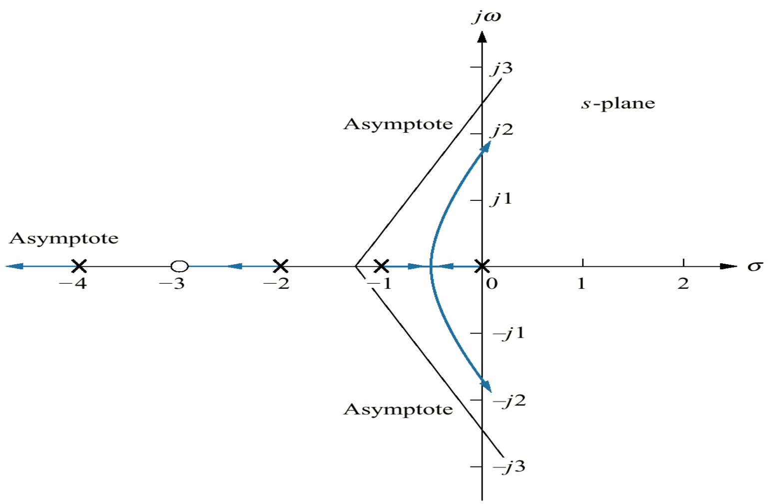

Figure 8.9
Root locus and asymptotes
for the system of Figure 8.8

REFINING THE SKETCH

- How to find accurately the points on the root locus and how to calculate the gain?
- Real - axis breakaway and break-in points
  - the point where the locus leaves the real axis, $-\sigma_1$, is called the **breakaway point**.
  - the point where the locus returns to the real axis, $+\sigma_2$, is called the **break-in point**.
  - At the breakaway or break-in point, the branches of the root locus form an angle of $180^\circ/n$ with the real axis, where n is the number of closed-loop poles arriving at or departing from the single breakaway or break-in point on the real axis (Kuo, 1991).

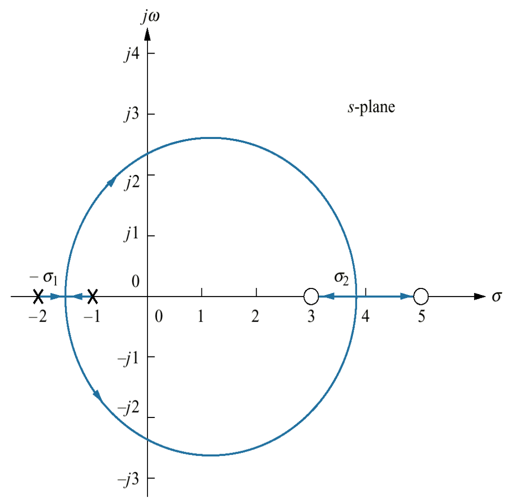

Figure 8.10
Root locus example showing real- axis breakaway ($-\sigma_1$) and break-in points (σ_2)

REFINING THE SKETCH

- The breakaway point occurs at a point of maximum gain on the real axis between the open–loop poles.
- The gain at the break-in point is the minimum gain on the real axis between the two zeros.
- The sketch in Figure 8.11 shows the variation of real-axis gain
  - The breakaway point is found at the maximum gain between -1 and -2
  - the break-in point is found at the minimum gain between +3 and +5

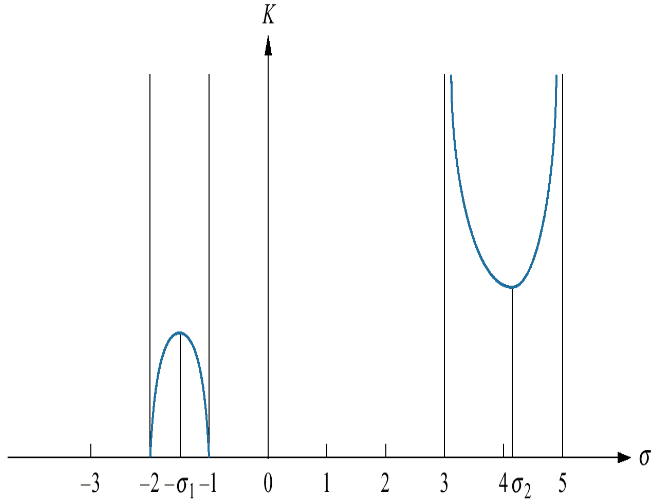

Figure 8.11
Variation of gain along the real axis for the root locus of Figure 8.10

REFINING THE SKETCH

- There are three methods for finding the points at which the root locus breaks away from and breaks into the real axis
  - The first method is to maximize and minimize the gain, K, using differential calculus

$$K = \frac{1}{G(s)H(s)} \quad (8.12)$$

- points along the real-axis segment of the root locus where breakaway and break-in points could exist $s = \sigma$

$$K = \frac{1}{G(\sigma)H(\sigma)} \quad (8.13)$$

- equation then represents a curve of K versus σ similar to that shown in Figure 8.11

- **Hence, if we differentiate Eq. (8.13) with respect to σ and set the derivative equal to zero,**

we can find the points of maximum/minimum gains and hence the breakaway and break-in points.

REFINING THE SKETCH

- Example 8.2
  - Find the breakaway and break-in points for the root locus of Figure 8.10, using differential calculus.
    - Using the open-loop poles and zeros, we represent the open-loop system whose root locus is shown in Figure 8.10 as follows:

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)} \tag{8.14}$$

But for all points along the root locus, $KG(s)H(s) = -1$, and along the real axis, $s = \sigma$
Hence,

$$\frac{K(\sigma^2 - 8\sigma + 15)}{(\sigma^2 + 3\sigma + 2)} = -1 \tag{8.15}$$

Solving for K, we find

$$K = \frac{-(\sigma^2 - 3\sigma + 2)}{(\sigma^2 + 8\sigma + 15)} \tag{8.16}$$

REFINING THE SKETCH

Differentiating K with respect to σ and setting the derivative equal to zero yields,

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2 - 26\sigma - 61)}{(\sigma^2 - 8\sigma - 15)^2} = 0 \qquad (8.17)$$

Solving for σ, we find σ =-1.45 and 3.82, which are the breakaway and break-in points.

REFINING THE SKETCH

- The second method is a variation on the differential calculus method: ***Transition Method***

$$\sum_{I}^{m} \frac{1}{\sigma + z_i} = \sum_{I}^{m} \frac{1}{\sigma + p_i} \qquad (8.18)$$

- Breakaway and break-in points satisfy the relationship where z_i and p_i are the negative of the zero and pole values, respectively, of $G(s)H(s)$.

- Solving Eq. (8.18) for σ, the real-axis values that minimize or maximize K, yields the breakaway and break-in points without differentiating

- The third method, the root locus program discussed in appendix G.2 at www.wiley.com/college/nise can be used to find the breakaway and break-in points
  - MATLAB also has the capability of finding breakaway and break-in points

Real axis value	Gain	
−1.41	0.008557	
−1.42	0.008585	
−1.43	0.008605	
−1.44	0.008617	
−1.45	0.008623	← Max. gain: breakaway
−1.46	0.008622	
3.3	44.686	
3.4	37.125	
3.5	33.000	
3.6	30.667	
3.7	29.440	
3.8	29.000	← Min. gain: break-in
3.9	29.202	

Table 8.2
Data for breakaway and break-in points for the root locus of Figure 8.10

REFINING THE SKETCH

- Example 8.3

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)}$$

  - Repeat Example 8.2 without differentiating
    - Using Eq.(8.18), (8.19)

$$\frac{1}{\sigma-3} + \frac{1}{\sigma-5} = \frac{1}{\sigma+1} + \frac{1}{\sigma+2}$$

Simplifying,

$$11\sigma^2 - 26\sigma - 61 = 0 \quad (8.20)$$

Hence, σ =-1.45 and 3.82, which agrees with Example 8.2

REFINING THE SKETCH

- The $j\omega$ -Axis crossing
  - The $j\omega$-axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation.
  - method of find the $j\omega$-axis crossing
    - ***Routh-Hurwitz criterion***

      - Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

    - The fact that at the $j\omega$ -axis crossing, the sum of angles from the finite open-loop poles and zeros must add to $(2k+1)180^\circ$.

REFINING THE SKETCH

❖ Example 8.4

- For the system of Figure 8.8, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable?
  - The closed-loop transfer function for the system of Figure 8.8 is

$$T(s)=\frac{K(s+3)}{s^4+7s^3+14s^2+(8+K)s+3K} \qquad (8.21)$$

Using the denominator and simplifying some of the entries by multiplying any row by a constant, we obtain the Routh array shown in Table 8.3

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\frac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

Table 8.3
Routh table for Eq. (8.21)

REFINING THE SKETCH

a complete row of zeros yields the possibility for imaginary axis roots.

$$-K^2 - 65K + 720 = 0 \qquad (8.22)$$

From this equation K is evaluated as

$$K = 9.65 \qquad (8.23)$$

Forming the even polynomial by using the s^2 row with $K = 9.65$, we obtain

$$(90 - K)s^2 + 21K = 80.35s^2 + 202.7 = 0 \qquad (8.24)$$

And s is found to be equal to $\pm j1.59$. Thus the root locus crosses the $j\omega$ -axis at $\pm j1.59$ at a gain of 9.65. We conclude that the system is stable for $0 \le K \le 9.65$

REFINING THE SKETCH

- Angles of departure and arrival
  - Root locus calculates the departure angle and the arrival angle from/to the complex poles and zeros.
  - The only unknown angle in the sum is the angle drawn from the pole that is $\in$ close

  - The Figure 8.15 calculate unknown angle

$$-\theta_1+\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2k+1)180^\circ \qquad (8.25a)$$

or

$$\theta_1=\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2k+1)180^\circ \qquad (8.25b)$$

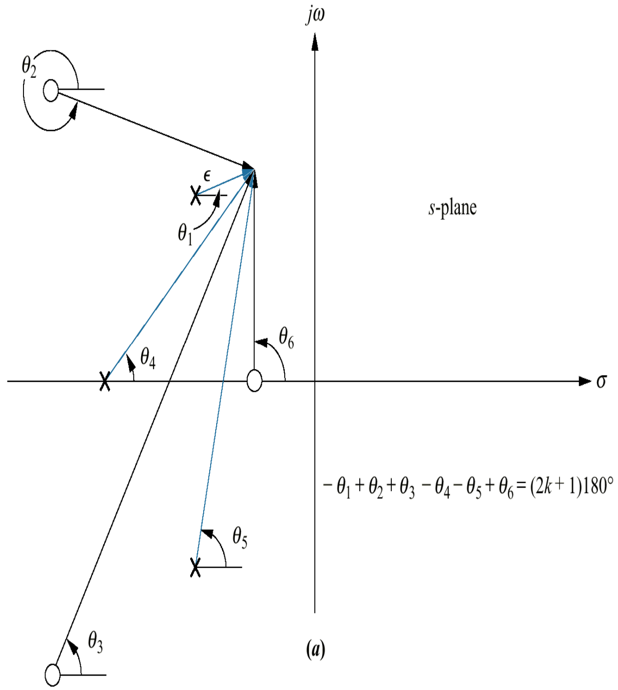

Figure 8.15
Open-loop poles and zeros and calculation of: angle of departure

REFINING THE SKETCH

- The only unknown angle in the sum is the angle drawn from the zeros that is ∈ close.

  The Figure 8.16 calculates unknown ang

$$-\theta_1+\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2k+1)180^\circ \qquad (8.26a)$$

$$\theta_2=\theta_1-\theta_3+\theta_4+\theta_5-\theta_6=(2k+1)180^\circ \qquad (8.26b)$$

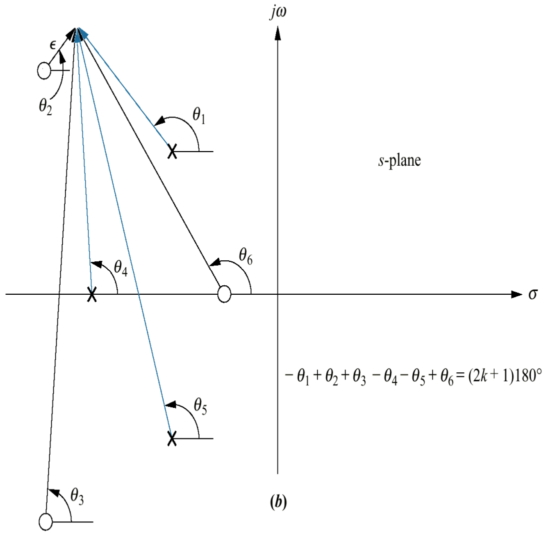

Figure 8.16
Open-loop poles and zeros and calculation of: angle of arrival

REFINING THE SKETCH

- Example 8.5
  - Given the unity feedback system of Figure 8.14 find the angle of departure from the complex poles and sketch the root locus

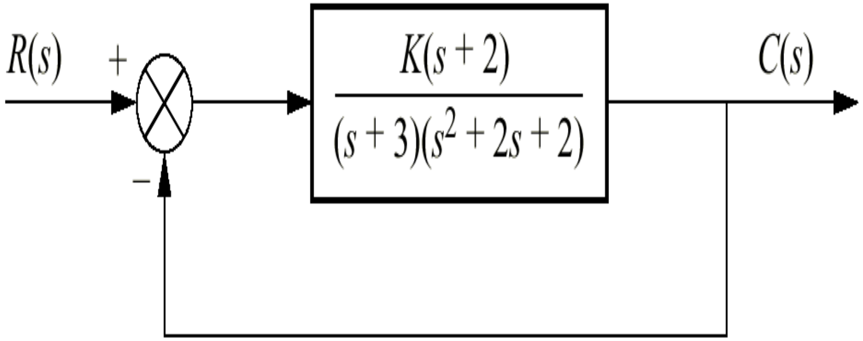

Figure 8.14
Unity feedback system with complex poles

- Using the poles and zeros of $G(s) = (s+2)/[(s+3)(s^2+2s+2)]$ as plotted in Figure 8.15, we calculate the sum of angles drawn to a point ∈ close to the complex pole, $-1+j1$, in the second quadrant. Thus,

REFINING THE SKETCH

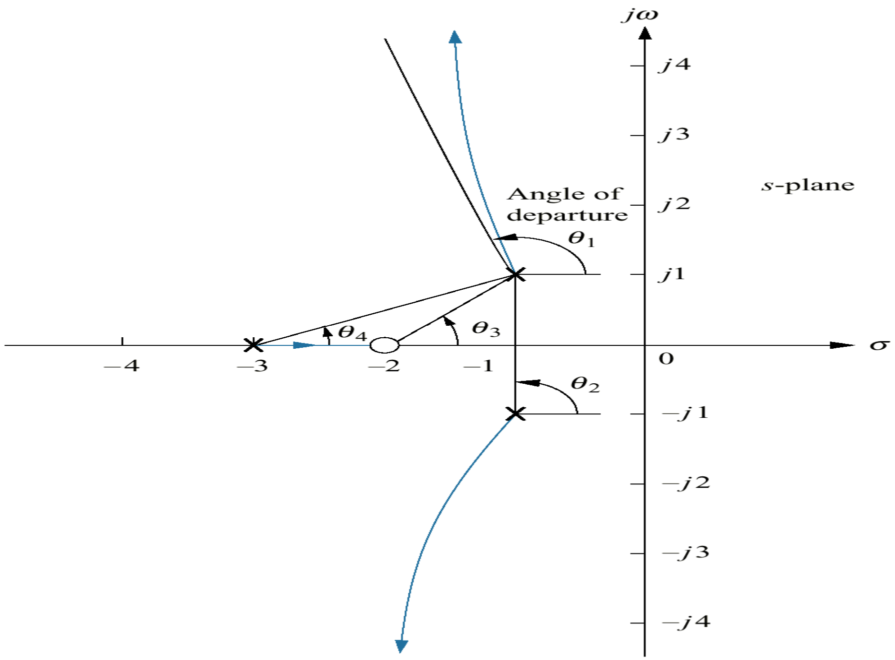

Figure 8.15
Root locus for system of
Figure 8.14 showing angle
of departure

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = -\theta_1 - 90^0 + \tan^{-1}(\frac{1}{1}) - \tan^{-1}(\frac{1}{2}) = 180^0 \qquad (8.27)$$

From which $\theta = -251.6^0 = 108.4^0$. A sketch of the root locus is show in Figure 8.15. Notice how the departure angle from the complex poles helps us to refine the shape

REFINING THE SKETCH

- Plotting and calibrating the root locus

Accurately locate points on the root

locus find their associated gain.

- Figure 8.18 shows the system's open-loop poles and zeros along with the ζ =0.45 line.
- Selecting the point at radius 2(r=2) on the ζ =0.45 line, add the angles of the zeros and subtract the angles of the poles

$$\theta_2 - \theta_1 - \theta_3 - \theta_4 - \theta_5 = -251.5^\circ \qquad (8.47)$$

The sum is not equal to an odd multiple of 180° , the point at radius =2 is not on the root locus.

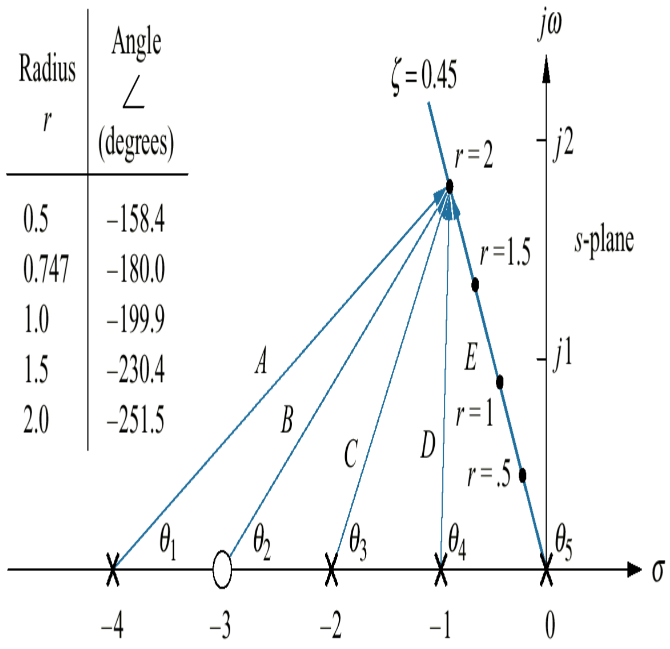

Figure 8.18
Finding and calibrating exact points on the root locus of Figure 8.12

REFINING THE SKETCH

- The table see that the point at radius 0.747 is on the root locus, since the angles add up to -180°.
- Gain K

$$K = \frac{|A||B||C||D|}{|B|} = 1.71 \qquad (8.48)$$

TRANSIENT RESPONSE DESIGN VIA GAIN ADJUSTMENT

- Conditions of second-order system approximation
  - Higher-order poles are much farther into the left half of the s-plane than the dominant second-order pair of poles.
  - Closed-loop zeros near the closed-loop second-order pole pair are nearly canceled by the close proximity of higher-order closed-loop poles.
  - Closed-loop zeros not canceled by the close proximity of higher-order closed-loop poles are far removed from the closed-loop poles.

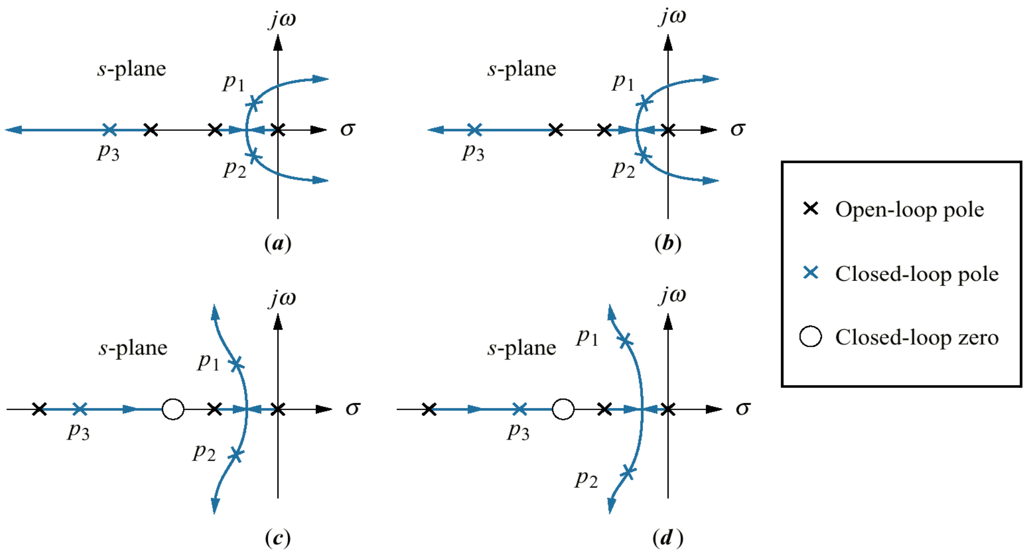

Figure 8.17
Making second-order approximations

TRANSIENT RESPONSE DESIGN VIA GAIN ADJUSTMENT

- Summarizing the design procedure for higher-order system
  - Sketch the root locus for the given system.
  - Assume the system is a second-order system without any zeros and then find the gain to meet the transient response specification.
  - Justify your second-order assumption by finding the location of all higher-order poles and evaluating the fact that they are much farther from the $j\omega$ than the times farther than the dominant second-order pair.

  Verify that closed-loop zeros are approximately canceled by higher-order poles.

  If closed-loop zeros are not canceled by higher-order closed-loop poles, be sure that the zero is far removed from the dominant second-order pole pair to yield approximately the same response obtained without the finite zero.
  - If the assumptions cannot be justified, your solution will have to be simulated in order to be sure it meets the transient response specification.

GENERALIZED ROOT LOCUS

- In figure 8.18, the parameter of interest is the open-loop pole at $-p_1$. How can we obtain a root locus for variations of the value of p_1?
- If the function $KG(s)H(s)$ is formed as

$$KG(s)H(s) = \frac{10}{(s+2)(s+p_1)} \tag{8.30}$$

  The problem is that p_1 is not a multiplying factor of the function, as the gain, K, was in all of the previous problems.

  - The close-loop transfer function's denominator is $1+KG(s)H(s)$, we effectively want to create an equivalent system whose denominator $1+p_1G(s)H(s)$.
  - For the system of Figure 8.18, the closed-loop transfer function is

$$T(s) = \frac{KG(s)}{1+KG(s)H(s)} = \frac{10}{s^2+(p_1+2)s+2p_1+10} \tag{8.31}$$

GENERALIZED ROOT LOCUS

- Isolating p_1,

$$T(s)=\frac{10}{s^2+2s+10+p_1(s+2)} \tag{8.32}$$

To convert the denominator to the form of K, let's divide by

$s^2+2s+10$ from the denominator and numerator.

$$T(s)=\frac{\dfrac{10}{s^2+2s+10}}{1+\dfrac{p_1(s+2)}{s^2+2s+10}} \tag{8.33}$$

- Conceptually, Eq. (8.33) implies that we have a system for which

$$KG(s)H(s)=\frac{p_1(s+2)}{s^2+2s+10} \tag{8.34}$$

GENERALIZED ROOT LOCUS

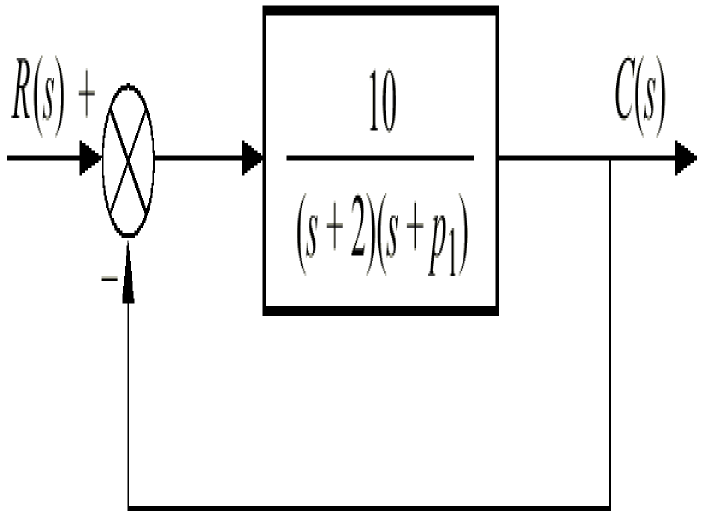

Figure 8.18
System requiring a root locus calibrated with p_1 as a parameter

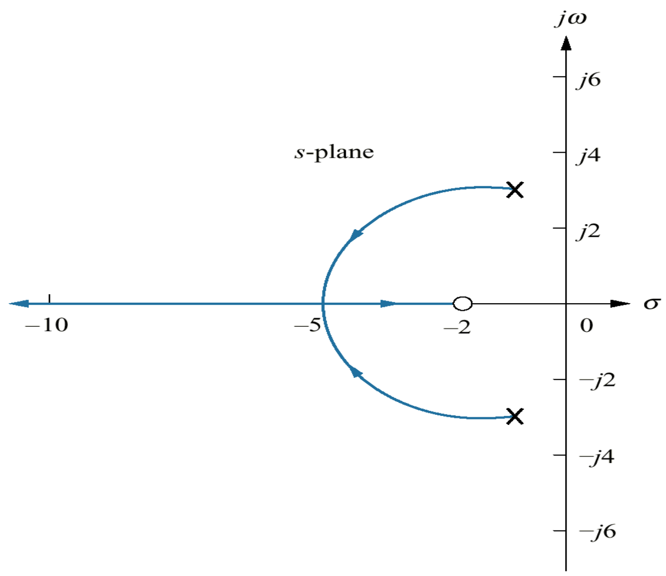

Figure 8.19
Root locus for the system of Figure 8.18, with p_1 as a parameter

Root Locus for Positive-Feedback Systems

- The properties of the root locus change dramatically if the feedback signal is added to the input rather than subtracted.
- The positive-feedback system show in Figure 8.20.

$$T(s) = \frac{KG(s)}{1 - KG(s)H(s)} \quad (8.35)$$

retrace the development of the root locus for the denominator of Eq.(8.35)

$$KG(s)H(s) = 1 = 1\angle k\,360^0 \quad k = 0, \pm 1, \pm 2, \pm 3, \cdots \quad (8.36)$$

- The root locus for positive-feedback systems consists of all points on the $s-$plane where the angle of $KG(s)H(s) = k360^\circ$.

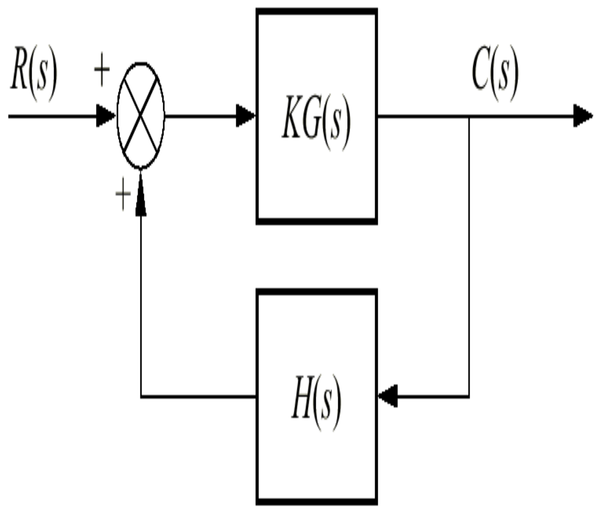

Figure 8.20
Positive-feedback system