Chapter 6.
STABILITY

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Introduction

- Three requirements for control systems:
  - Transient response (Chapter. 4, 8)
  - Stability (Chapter. 6)
  - Steady-state errors (Chapter. 7)

- Total response of a system: $c(t) = c_{forced}(t) + c_{natural}(t) \qquad (6.1)$
  - A linear, time-invariant system is stable if the natural response approaches zero as time approaches infinity.
  - A linear time-invariant system is unstable if the natural response grows without bound as time approaches infinity.
  - A linear time-invariant system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates time approaches infinity.

Introduction

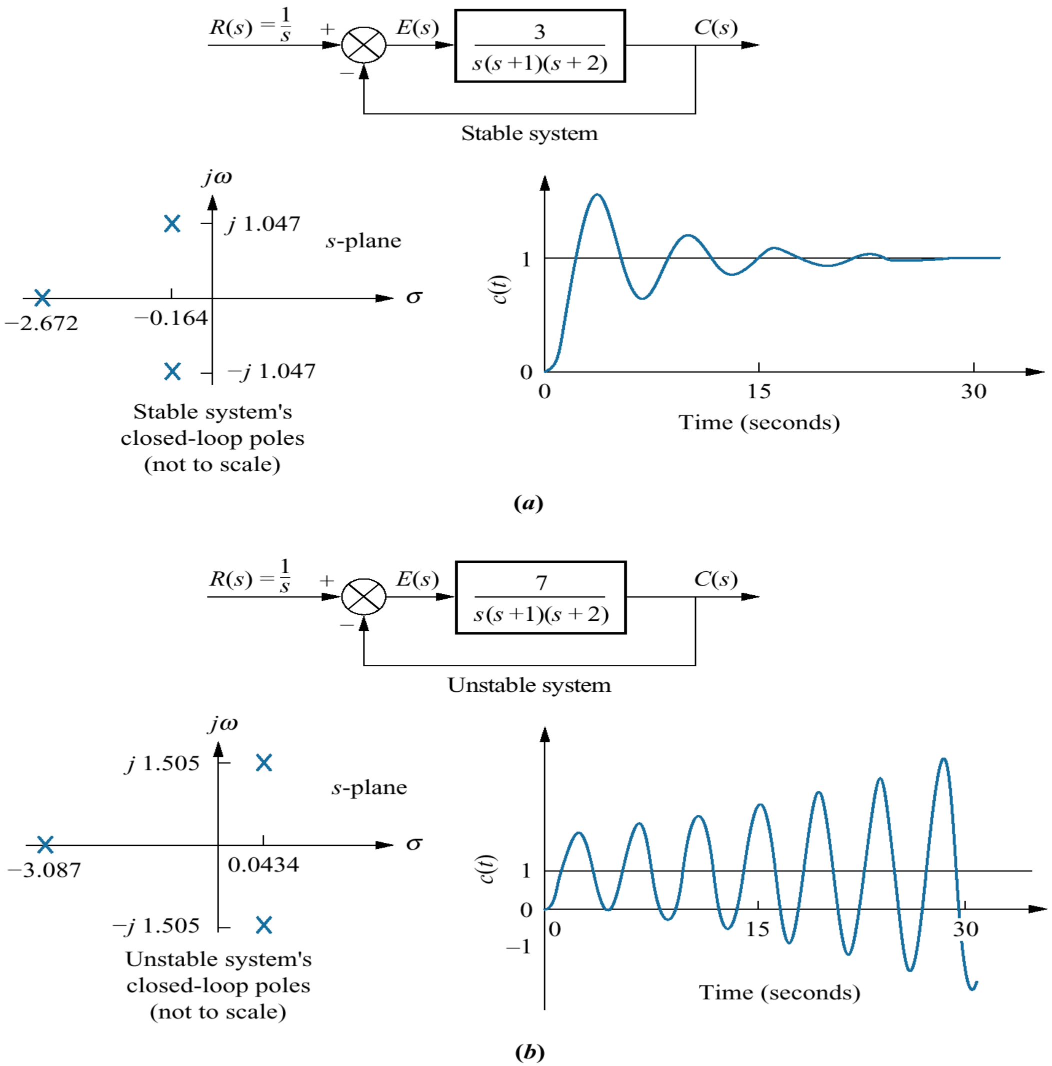


Figure 6.1
Closed-loop poles and response:
a. stable system;
b. unstable system

- Unstable systems have closed-loop transfer functions with **at least one pole in the right half-plane** and/or poles of **multiplicity greater than 1 on the imaginary axis.**
- Marginally stable systems have closed-loop transfer functions with only **imaginary axis poles of multiplicity 1** and poles in the left half-plane.

Introduction

- Common cause of problems in finding closed-loop poles:

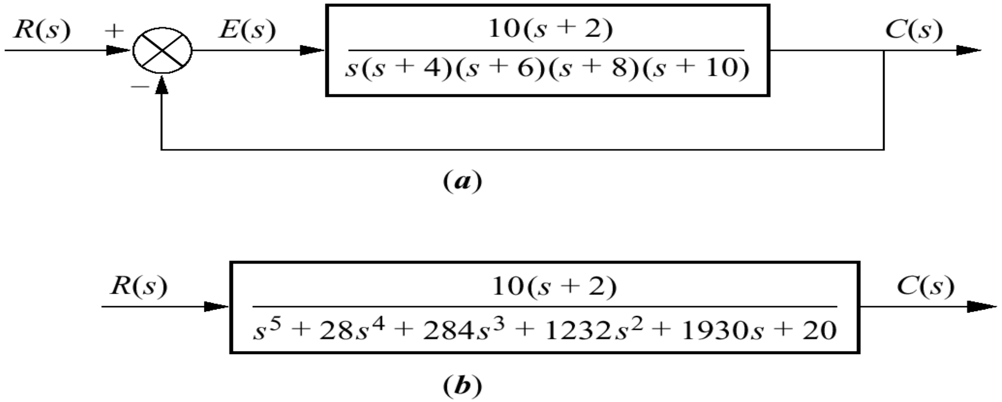
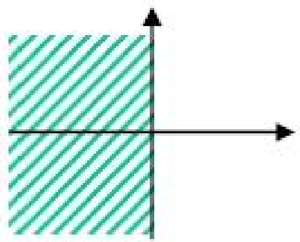


Figure 6.2
Common cause of problems in finding closed-loop poles:
a. original system;
b. equivalent system

- Stability of a system: If the closed-loop transfer function has only **left-half-plane poles**, then the system is stable.

- a_i: real and positive, or complex with a positive real part

$$P(s) = \sum_{i=1}^{n} (s + a_i)$$



Routh-Hurwitz Criterion

- Definition of Routh-Hurwitz Criterion
  - In the closed-loop system,
    How many poles are in the left half-plane, in the right half-plane, and in the $j\omega$ -axis.
  - How many can be solved but where are they is not solved.
  - **Routh Stability Test**
  - The method requires two steps:
    (1) Generating a basic Routh Table
    (2) Interpreting the basic Routh Table

Routh-Hurwitz Criterion

- Generating a basic Routh table

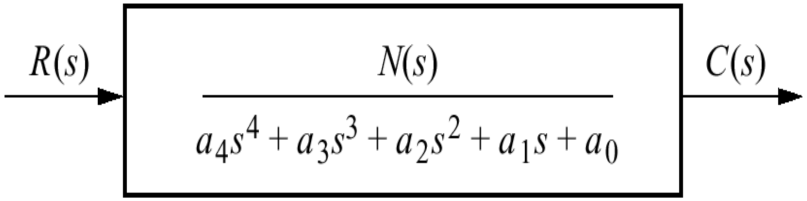


Figure 6.3
Equivalent closed-loop transfer function

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Table 6.1
Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Table 6.2
Completed Routh table

Routh-Hurwitz Criterion

- EX: 6.1
  - Make the Routh table for the system shown in Figure 6.4(a).

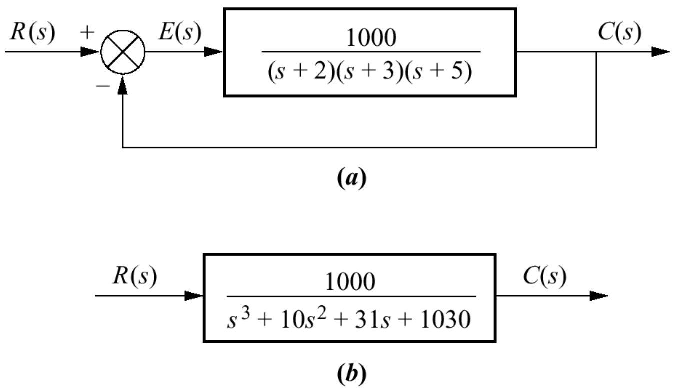


Figure 6.4
a. Feedback system for Example 6.1;
b. Equivalent closed-loop system

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\frac{-\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1} = -72$	$\frac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1} = 0$
s^0	$\frac{-\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72} = 103$	$\frac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$	$\frac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$

Table 6.3
Completed Routh table for Example 6.1

Routh-Hurwitz Criterion

- Interpreting the basic Routh table
  - The number of roots of the polynomial that are in the RHP is equal to the number of sign changes in the first column.
  - **A system is stable if there are no sign changes in the first column of the Routh Table.**

Routh-Hurwitz Criterion: Special Cases

❖ Zero only in the first column of a row

- An epsilon, $\mathcal{E}$ is assigned to replace the zero in the first column.
- $\mathcal{E}$ is allowed to approach zero from either the positive or negative side.

- Entire row that consists of zeros
  - Even or odd polynomial that is a factor of the original polynomial.
  - Even polynomials only have roots that are symmetrical about the origin.
  - The row previous to the row of zeros contains the **even polynomial that is a factor of the original polynomial.**
  - Everything from the row containing the even polynomial down to the end of the Routh table is a test of only the even polynomial.

Routh-Hurwitz Criterion: Special Cases

❖ EX: 6.2

- Determine the stability of the closed-loop transfer function.

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \qquad (6.2)$$

- If ε is a chosen positive and there are two sign changes, then the system is unstable and has two poles in the right half-plane.
- If ε is chosen negative and there are two sign changes, then the system is unstable, with two poles in the right half-plane.

Table 6.4
Completed Routh table for Example 6.2

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Table 6.5
Determining signs in first column of a Routh table with zero as first element in a row

Label	First Column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

Routh-Hurwitz Criterion: Special Cases

❖ EX: 6.3

- Determine the stability of the closed-loop transfer function.

$$T(s)=\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3} \qquad (6.6)$$

- First write a polynomial that has the reciprocal roots of the denominator of Eq. (6.6).

$$D(s)=3s^5+5s^4+6s^3+3s^2+2s+1 \qquad (6.7)$$

- We form the Routh table as shown in Table 6.6 using Eq.(6.7). There are two sign changes, the system is unstable and has two right-half-plane poles.

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

Table 6.6
Routh table for Example 6.3

Routh-Hurwitz Criterion: Special Cases

❖ EX: 6.4

- Determine the number of right-half-plane poles in the closed-loop transfer function.

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56} \qquad (6.8)$$

- At the second row we multiply through by 1/7 for convenience.

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Table 6.7
Routh table for Example 6.4

$$P(s) = s^4 + 6s^2 + 8 \qquad (6.9)$$

- Differentiate the polynomial with respect to s and obtain.
- There are no right-half-plane poles.

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0 \qquad (6.10)$$

Routh-Hurwitz Criterion: Special Cases

❖ EX: 6.5

- Tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega - a$xis

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20} \quad (6.11)$$

- For convenience the s^6 row is multiplied by 1/10, and the s^5 row is multiplied by 1/20. At the s^3 row, we obtain a row of zeros. Moving back one row to s^4 , we extract the even polynomial, P(s), as

$$P(s) = s^4 + 3s^2 + 2 \quad (6.12)$$

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	$-$~~10~~ -1	$-$~~20~~ -2	~~10~~ 1	~~20~~ 2	0
s^5	~~20~~ 1	~~60~~ 3	~~40~~ 2	0	0
s^4	1	3	2	0	0
s^3	~~0~~ ~~4~~ 2	~~0~~ ~~6~~ 3	~~0~~ ~~0~~ 0	0	0
s^2	~~$\frac{3}{2}$~~ 3	~~2~~ 4	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

Table 6.8
Routh table for Example 6.5

Routh-Hurwitz Criterion: Special Cases

- Taking the derivative with respect to s to obtain the coefficients that replace the row of zeros in the s^3 row, we find

$$\frac{dP(s)}{ds} = 4s^3 + 6s + 0 \qquad (6.13)$$

- Replace the row of zeros with 4, 6, and 0 and multiply the row by ½ for convenience. No sign changes exist from s^4 row down to s^0 row. The even polynomial does not have right-half-plane poles. Since there are no right-half-plane poles, no left-half-plane poles are present because of the requirement for symmetry.

Table 6.9 Summary of pole locations for Example 6.5

	Polynomial		
Location	**Even (fourth-order)**	**Other (fourth-order)**	**Total (eighth-order)**
Right half-plane	0	2	2
Left half-plane	0	2	2
$j\omega$	4	0	4

- There are two sign changes. The system has two poles in the right half-plane, two poles in the left half-plane, and four poles on the $j\omega$-axis. It is unstable because of the right-half-plane poles.

Routh-Hurwitz Criterion: Additional Examples

- EX: 6.6
  - Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega - a\text{xis}$ for the system of Figure 6.6

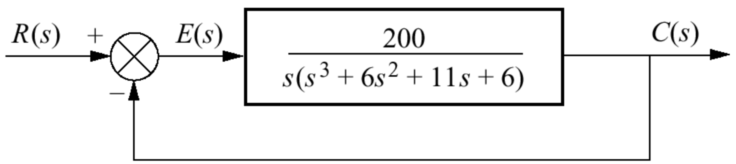


Figure 6.6 Feedback control system for Example 6.6

  - First find the closed-loop transfer function as

$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200} \qquad (6.14)$$

  - At the S^1 row, there is a negative coefficient. Thus, there are two sign changes. The system is unstable, since it has two right-half-plane poles and two left-half-plane poles. The system can't have $j\omega$ poles since a row of zeros didn't appear in the Routh table.

s^4	1	11	200
s^3	~~6~~ 1	~~6~~ 1	
s^2	~~10~~ 1	~~200~~ 20	
s^1	-19		
s^0	20		

Table 6.10 Routh table for Example 6.6

Routh-Hurwitz Criterion: Additional Examples

- EX: 6.7
  - Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega - a$xis for the system of Figure 6.7

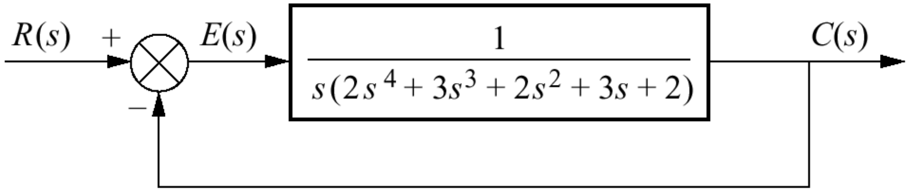


Figure 6.6 Feedback control system for Example 6.7

  - The closed-loop transfer function is

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1} \qquad (6.15)$$

  - The entire row is not zero, simply replace the zero with a small quantity, $\in$, and continue the table, permitting $\in$ to be a small, positive quantity. We find that the first term of the s^2 row is negative. Thus, there are two sign changes, and the system is unstable, with two poles in the right half-plane. The remaining poles ate in the left half-plane.

Routh-Hurwitz Criterion: Additional Examples

s^5	2	2	2
s^4	3	3	1
s^3	$\not{0}$ ϵ	$\frac{4}{3}$	
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$		
s^0	1		

Table 6.11
Routh table for Example 6.7

- Using the denominator of Eq.(6.15), we form a polynominal by writing the coefficients in reverse order.

$$s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2 \qquad (6.16)$$

- In this case, we also produce a zero only in the first column at the s^2 row.

The system is unstable.

s^5	1	3	3
s^4	2	2	2
s^3	2	2	
s^2	$\not{0}$ ϵ	2	
s^1	$\frac{2\epsilon - 4}{\epsilon}$		
s^0	2		

Table 6.12
Alternative Routh table for Example 6.7

Routh-Hurwitz Criterion: Additional Examples

- EX: 6.8
  - Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega-a$xis for the system of Figure 6.8. Draw conclusions about the stability of the closed-loop system.

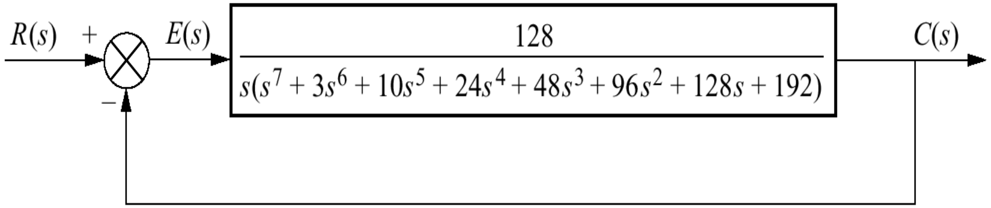


Figure 6.8 Feedback control system for Example 6.8

  - The closed-loop transfer function is

$$T(s) = \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128} \qquad (6.17)$$

  - Return to the s^6 row and form the even polynomial:

$$P(s) = s^6 + 8s^4 + 32s^2 + 64 \qquad (6.18)$$

  - Differentiate this polynomial with respect to s to form the coefficients that will replace the row of zeros:

$$\frac{dP(s)}{ds} = 6s^5 + 32s^3 + 64s + 0 \qquad (6.19)$$

Routh-Hurwitz Criterion: Additional Examples

s^8	1	10	48	128	128
s^7	~~3~~ 1	~~24~~ 8	~~96~~ 32	~~192~~ 64	
s^6	~~2~~ 1	~~16~~ 8	~~64~~ 32	~~128~~ 64	
s^5	~~0~~ ~~6~~ 3	~~0~~ ~~32~~ 16	~~0~~ ~~64~~ 32	~~0~~ ~~0~~ 0	
s^4	~~$\frac{8}{3}$~~ 1	~~$\frac{64}{3}$~~ 8	~~64~~ 24		
s^3	~~−8~~ −1	~~−40~~ −5			
s^2	~~3~~ 1	~~24~~ 8			
s^1	3				
s^0	8				

Table 6.13
Routh table for Example 6.8

- Replace the row of zeros at the s^5 row by the coefficients of Eq.(6.19) and multiply through by ½ for convenience. Then complete the table.

Table 6.14 Summary of pole locations for Example 6.8

Location	**Polynomial** **Even (sixth-order)**	**Other (second-order)**	**Total (eighth-order)**
Right half-plane	2	0	2
Left half-plane	2	2	4
$j\omega$	2	0	2

Note: rhp = right half-plane, lhp = left half-plane.

- The system has two poles in the right half-plane, four poles in the left half-plane, and two poles on the $j\omega - a\text{xis}$, which are of unit multiplicity. The closed-loop system is unstable because of the right-half plane poles.

Routh-Hurwitz Criterion: Additional Examples

- EX: 6.9
  - Find the range of gain, K, for the system of Figure 6.10 that will cause the system to be stable, unstable, and marginally stable. Assume K > 0.

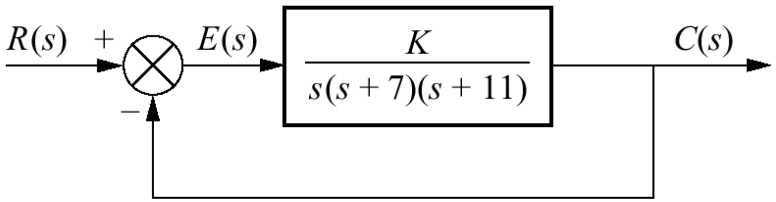


Figure 6.10 Feedback control system for Example 6.9

  - First find the closed-loop transfer function as

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K} \qquad (6.20)$$

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

Table 6.15 Routh table for Example 6.9

  - If K < 1386, all terms in the first column will be positive, and since there is no sign change, the system will have three poles in the left half-plane and be stable.

Routh-Hurwitz Criterion: Additional Examples

- If K > 1386, the system has two right-half-plane poles and one left-half-plane pole, which makes the system unstable.
- If K = 1386, returning to the s^2 row and replacing K with 1386, we form the even polynomial

$$P(s) = 18s^2 + 1386 \qquad (6.21)$$

- Differentiating with respect to s, we have

$$\frac{dP(s)}{ds} = 36s + 0 \qquad (6.22)$$

s^3	1	77
s^2	18	1386
s^1	~~0~~ 36	
s^0	1386	

Table 6.16
Routh table for Example 6.9 with K = 1386

- There is no sign change above the even polynomial, the remaining root is in the left half-plane. Therefore the system is marginally stable.

Routh-Hurwitz Criterion: Additional Examples

- EX: 6.10
  - Factor the polynomial

$$s^4 + 3s^3 + 30s^2 + 30s + 200 \qquad (6.23)$$

  - Form the even polynomial at the s^2 row:

$$P(s) = s^2 + 10 \qquad (6.24)$$

s^4	1	30	200
s^3	~~3~~ 1	~~30~~ 10	
s^2	~~20~~ 1	~~200~~ 10	
s^1	~~0~~ 2	~~0~~ 0	
s^0	10		

Table 6.17
Routh table for Example 6.10

  - Dividing Eq. (6.23) by (6.24) yields $(s^2 + 3s + 20)$ as the other factor.

$$\begin{aligned} s^4 + 3s^3 + 30s^2 + 30s + 200 &= (s^2 + 10)(s^2 + 3s + 20) \\ &= (s + j3.1623)(s - j3.1623) \qquad (6.25) \\ &\quad \times (s + 1.5 + j4.213)(s + 1.5 - j4.213) \end{aligned}$$

Stability in State Space

- Stability in State Space
  - The eigenvalues of the matrix A are solutions of the equation $\det(sI - A) = 0$.
  - The eigenvalues of a matrix A, are values of λ that permit a nontrivial solution (other than 0) of x in the equation.

$$\mathrm{Ax}=\lambda \mathrm{x} \qquad (6.26)$$

$$\lambda \mathrm{x}\text{-}\mathrm{Ax}=0 \qquad (6.27) \quad \text{or} \quad (\lambda \mathrm{I}\text{-}\mathrm{A})\mathrm{x}=0 \qquad (6.28)$$

$$\mathrm{x}=(\lambda \mathrm{I}\text{-}\mathrm{A})^{-1}0 \qquad (6.29) \quad \text{or} \quad \mathrm{x}=\frac{\mathrm{adj}(\lambda \mathrm{I}\text{-}\mathrm{A})}{\det(\lambda \mathrm{I}\text{-}\mathrm{A})}0 \qquad (6.30)$$

  - The value of λ are calculated by forcing the denominator to zero: $\det(\lambda \mathrm{I}\text{-}\mathrm{A})=0 \qquad (6.31)$
  - The system transfer function has $\det(\mathrm{sI}\text{-}\mathrm{A})$ in the denominator because of the presence of $(\mathrm{sI}\text{-}\mathrm{A})^{-1}$. thus,

$$\det(\mathrm{sI}\text{-}\mathrm{A})=0 \qquad (6.32)$$

Stability in State Space

- EX: 6.11
  - Find out how many poles are in the left-plane, in the right half-plane, and on the $j\omega$ -axis.

$$\dot{x}=\begin{bmatrix}0 & 3 & 1\\ 2 & 8 & 1\\ -10 & -5 & -2\end{bmatrix}x+\begin{bmatrix}10\\ 0\\ 0\end{bmatrix}u, \quad (6.33a)$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}x \quad (6.33b)$$

  - First form $(sI-A)$:

$$(sI-A)=\begin{bmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix}-\begin{bmatrix}0 & 3 & 1\\ 2 & 8 & 1\\ -10 & -5 & -2\end{bmatrix}=\begin{bmatrix}s & -3 & -1\\ -2 & s-8 & -1\\ 10 & 5 & s+2\end{bmatrix} \quad (6.34)$$

  - Now find the $\det(sI-A)$: $\det(sI-A)=s^3-6s^2-7s-52 \quad (6.35)$
  - Using this polynomial, form the Routh table of Table 6.18.

s^3	1	-7
s^2	$\not{-6}$ -3	$\not{-52}$ -26
s^1	$\not{-\frac{47}{3}}$ -1	$\not{0}$ 0
s^0	-26	

Table 6.18
Routh table for Example 6.11